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Worldline path integrals for a Dirac particle in a weak gravitational plane wave

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Abstract. The problem of a relativistic spinning particle interacting with a weak gravitational plane wave in $(3+1)$ dimensions is formulated in the frame work of covariant supersymmetric path integrals. The relative Green function is expressed through a functional integral over bosonic trajectories that describe the external motion and fermionic variables that describe the spin degrees of freedom. The $(3+1)$ dimensional problem is reduced to the $(1+1)$ dimensional one by using an identity. Next, the relative propagator is exactly calculated and the wave functions are extracted.

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1 Introduction

As is well known, Feynman has introduced his famous path integral quantization method in order to satisfy the need of comprehension of quantum mechanics [1]. Many problems in nonrelativistic quantum mechanics are exactly solved by the use of the path integral approach starting from their classical origins (i.e. classical actions). Moreover, the path integral remains a useful quantization procedure mainly when it becomes, like in cosmology, difficult to use other methods [2, 3]. This explains the increased interest in developing path integration techniques [4–8].

In relativistic quantum mechanics, especially, for the Dirac equation the Feynman method has not known the same development, mainly because of the fact that the spin has no classical origin and the difficulty of inserting the anticommuting γ-matrices by means of paths. However, a successful supersymmetric formulation for relativistic spinning particles was elaborated by Fradkin and Gitman [9] according to the Feynman standard form

$$\sum_{\text{paths}} \exp \mathrm{i}S(\text{path})\,, \tag{1}$$

where S is a supersymmetric action that describes at the same time the external motion and the internal one, related to the spin of the particle. Elsewhere, the same problem is reconsidered following the so-called global and local representations by Alexandrou et al. [10]. We notice, also, that the Fradkin–Gitman formulation is generalized to the case of arbitrary dimensions in [11] and to the case of the Dirac equation with torsion field in [12]. Also, the supersymmetric path integrals are used to solve the problem of a spin half particle subject to pseudoscalar potentials [13, 14].

Recently, the problem of a massive relativistic particle in the background of a weak gravitational plane wave has been considered in [15, 16]. The corresponding Green functions for both the spinless and spin-$\frac{1}{2}$ cases are obtained by alternative methods. The considered situation of the linear approximation of gravity simplifies the problem of relativistic particles in external fields and enables the authors to avoid complications related to the coupling with the gravitational field. Moreover, the subject of gravitational waves has more significant implications in electrodynamics [17]. Namely, if an electromagnetic wave propagates in a vacuum in which a gravitational wave is propagating in the same direction, new electromagnetic waves appear at the sum and difference frequencies [18]. Consequently, the detection of waves at the sum and difference frequencies might be used as a method for detecting gravitational waves. In this context the problem of a charged spin half particle subject to a weak gravitational plane wave remains an important topic in theoretical and experimental physics.

In the present paper, we propose a straightforward method for solving the problem of a Dirac particle subject to the background of a weak gravitational plane wave by the use of supersymmetric path integrals, which proved most fruitful in finding analytical and exact expressions of the wave functions and the energy spectrum of the fermion. In the first stage we generalize the path integral formulation given by Fradkin and Gitman following the global projection. Then by incorporating an identity, the $(3+1)$ dimensional problem will be reduced to a $(1+1)$ di-

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mensional problem. Next, we show, after integrating over odd trajectories, that the relative Green function can be expressed only through bosonic path integrals that will then be straightforward. Finally, we easily extract the wave functions.

2 Path integral formulation

The linearization of the Einstein equations of general relativity is the natural way to describe the interaction of matter with a weak gravitational field. In this approximation the metric $g_{\mu\nu}(x)$ that describes the gravitational field is considered as a small perturbation on the flat Minkowski metric $\eta_{\mu\nu}$ [19],

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \tag{2}$$

and

$$g^{\mu\nu}(x) = \eta^{\mu\nu} - h^{\mu\nu}(x)\,, \tag{3}$$

where the terms of order $h^2, h^3, \ldots$ are systematically discarded ($h_{\mu\nu} \ll 1$).

Also, to obtain the solution of the linearized Einstein equations, it is convenient to work in the harmonic coordinate gauge [20]

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0\,, \tag{4}$$

which reduces in the weak field approximation to the so-called Lorentz gauge,

$$\partial_\lambda h^\lambda_\mu = \frac{1}{2}\partial_\mu h\,, \tag{5}$$

where

$$h(x) = \eta^{\mu\nu}h_{\mu\nu}(x)\,. \tag{6}$$

Then, the plane wave solution is given by

$$h_{\mu\nu}(k\cdot x) = a_{\mu\nu}f(k\cdot x)\,, \tag{7}$$

where f is an arbitrary function of the variable $(k_\mu x^\mu)$, k being the wave propagation vector (with $k^2 = 0$).

Besides the Lorentz gauge condition

$$k^\mu h_{\mu\nu}(k\cdot x) = k^\nu h_{\mu\nu}(k\cdot x) = 0\,, \tag{8}$$

we assume that $h(x) = 0$ and the tensor $h_{\mu\nu}$ is symmetric

$$h_{\mu\nu}(k\cdot x) = h_{\nu\mu}(k\cdot x)\,. \tag{9}$$

In this case the Dirac equation can be written as

$$\left[\gamma^\mu\left(P_\mu - \frac{1}{2}h_{\mu\nu}(k\cdot x)P^\nu\right) - m\right]\psi(x) = 0\,, \tag{10}$$

and the corresponding Green function $S^c(x_b, x_a)$ is a solution of the equation

$$\left[\gamma^\mu\left(P_\mu - \frac{1}{2}h_{\mu\nu}(k\cdot x)P^\nu\right) - m\right]S(x, x') = -\delta(x - x')\,. \tag{11}$$

Let us construct a global path integral representation starting from (11). It is known that $S^c(x_b, x_a)$ can be represented as a matrix element of an operator $\mathbb{S}^c$

$$S^c(x_b, x_a) = \langle x_b|\,\mathbb{S}^c\,|x_a\rangle\,, \tag{12}$$

where

$$\mathbb{S}^c = \frac{-1}{K_-} = -K_+\frac{1}{K_-K_+}\,, \tag{13}$$

and the operators K_- and K_+ are given by

$$K_\pm = \gamma^\mu\left(P_\mu - \frac{1}{2}h_{\mu\nu}(k\cdot x)P^\nu\right) \pm m\,. \tag{14}$$

Note that we use this procedure to obtain a Bose-type operator that has a quadratic form with respect to the γ-matrices and to avoid the usual five-dimensional extension (i.e. without the use of the matrix γ^5 employed in [9, 11]).

Taking into account that $[\gamma^\mu, \gamma^\nu]_+ = \eta^{\mu\nu}$, and by using the properties (4) the product K_-K_+ can be rearranged as follows:

$$\begin{aligned} K_-K_+ = {} & P^2 - h^{\mu\nu}(k\cdot x)P_\mu P_\nu \\ & - \frac{\mathrm{i}}{2}f'(k\cdot x)\,k^\mu a^{\nu\sigma}P_\sigma\gamma^\mu\gamma^\nu - m^2\,. \end{aligned} \tag{15}$$

Now, in order to build a global representation we use the relation $\int \mathrm{d}x\,|x\rangle\langle x| = 1$. We get

$$S^c(x_b, x_a) = \left[\gamma^\mu\left(P_\mu - \frac{1}{2}h_{\mu\nu}(k\cdot x)P^\nu\right) + m\right]G^c(x_b, x_a)\,. \tag{16}$$

Here, the Green function $G^c(x_b, x_a)$ that we suggest to calculate via path integration has the following proper time representation:

$$G^c(x_b, x_a) = \mathrm{i}\int_0^\infty \mathrm{d}\lambda\,\langle x_b|\exp(-\mathrm{i}\mathcal{H}(\lambda))\,|x_a\rangle\,, \tag{17}$$

with

$$\begin{aligned} \mathcal{H}(\lambda) = \lambda\Bigg(& -P^2 + h^{\mu\nu}(k\cdot x)P_\mu P_\nu \\ & + \frac{\mathrm{i}}{2}f'(k\cdot x)\,k^\mu a^{\nu\sigma}P_\sigma\gamma^\mu\gamma^\nu + m^2\Bigg)\,, \end{aligned} \tag{18}$$

and the operator $\left[\gamma^\mu\left(P_\mu - \frac{1}{2}h_{\mu\nu}(k\cdot x)P^\nu\right) + m\right]$ will eliminate the superfluous states caused by the product K_-K_+ in (13).

According to the Fradkin–Gitman formulation the Green function can be represented by means of supersymmetric path integrals as follows:

$$G^c = \exp\left(\mathrm{i}\gamma^\mu \frac{\partial_l}{\partial\theta^\mu}\right)\int_0^\infty \mathrm{d}\lambda_0 \mathrm{e}^{-\mathrm{i}m^2\lambda} \times \int \mathrm{D}x \int \mathrm{D}p \int \mathrm{D}\lambda \int \mathrm{D}\pi \int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi \times \exp\left\{\mathrm{i}\int_0^1 \mathrm{d}\tau \left[\lambda\left(p^2 - h^{\mu\nu}(k\cdot x)p_\mu p_\nu\right) + p_\sigma\left(\dot{x}^\sigma - 2\mathrm{i}\lambda f'(k\cdot x)\, k^\mu a^{\nu\sigma}\psi^\mu\psi^\nu\right) - \mathrm{i}\psi_\mu\dot{\psi}^\mu + \pi\dot{\lambda}\right] + \psi_\mu(1)\,\psi^\mu(0)\right\}\Bigg|_{\theta=0}, \tag{19}$$

where the measure $\mathcal{D}\psi$ is given by

$$\mathcal{D}\psi = \mathrm{D}\psi\left[\int_{\psi(0)+\psi(1)=0} \mathrm{D}\psi \exp\left\{\int_0^1 \psi_\mu\dot{\psi}^\mu \mathrm{d}\tau\right\}\right]^{-1} \tag{20}$$

and the θ^μ and ψ^μ are odd variables, anticommuting with the γ-matrices.

We notice that, integrating over the momenta and separating the gauge-fixing term $\pi\dot{\lambda}$ and the boundary term $\psi_n(1)\,\psi^n(0)$, we obtain the super-gauge invariant action

$$\mathcal{A} = \int_0^1 \mathrm{d}\tau\left[-\frac{1}{4\lambda}\left(\dot{x}^2 + h_{\mu\nu}(k\cdot x)\,\dot{x}^\mu\dot{x}^\nu\right) - \mathrm{i}\psi_\mu\dot{\psi}^\mu + \mathrm{i}k_\mu f'(k\cdot x)\, a_{\nu\alpha}\dot{x}^\alpha\psi^\mu\psi^\nu\right], \tag{21}$$

which resembles the Berezin–Marinov action [21–24].

Note that the Lagrangian given in [15] leads to a pseudoclassical action that differs from the present one by the five-dimensional extension. We show in the next section that by summing over all possible pseudoclassical paths we obtain the same wave functions.

3 Exact solution

To begin with, let us fix in (19) the gauge over the proper time λ by performing the functional integral over π and λ; the Green function will take the following form:

$$G^c = \exp\left(\mathrm{i}\gamma^\mu \frac{\partial_l}{\partial\theta^\mu}\right)\int_0^\infty \mathrm{d}T \mathrm{e}^{-\mathrm{i}m^2 T} \times \int \mathrm{D}x \int \mathrm{D}p \int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi \times \exp\left\{\mathrm{i}\int_0^T \mathrm{d}\tau \left[p^2 - h^{\mu\nu}(k\cdot x)p_\mu p_\nu + p_\sigma\left(\dot{x}^\sigma - 2\mathrm{i}f'(k\cdot x)\, k^\mu a^{\nu\sigma}\psi^\mu\psi^\nu\right) - \mathrm{i}\psi_\mu\dot{\psi}^\mu\right] + \psi_\mu(1)\,\psi^\mu(0)\right\}\Bigg|_{\theta=0}. \tag{22}$$

Then, in order to reduce the $(3+1)$ dimensional problem to a $(1+1)$ dimensional one and to obtain the closed expression of the Green function, we introduce the usual identity

$$\int \mathrm{d}\phi_a\,\mathrm{d}\phi_b\delta\left(\phi_a - k\cdot x(0)\right) \times \int \mathrm{D}\phi \mathrm{D}p_\phi \exp\left\{\mathrm{i}\int_0^T \mathrm{d}\tau p_\phi\left(\dot{\phi} - k\dot{x}\right)\right\} = 1\,. \tag{23}$$

We get

$$G^c = \exp\left(\mathrm{i}\gamma^n \frac{\partial_l}{\partial\theta^n}\right)\int \mathrm{d}\phi_a\,\mathrm{d}\phi_b\delta\left(\phi_a - k\cdot x(0)\right)\int_0^\infty \mathrm{d}T \times \int \mathrm{D}\phi \mathrm{D}p_\phi \int \mathrm{D}x \int \mathrm{D}p \int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi \times \exp\left\{\mathrm{i}\int_0^T \mathrm{d}\tau\left[p^2 - h^{\mu\nu}(\phi)p_\mu p_\nu - m^2 - 2\mathrm{i}f'(\phi)\, k^\mu a^{\nu\sigma}p_\sigma\psi^\mu\psi^\nu - \mathrm{i}\psi_n\dot{\psi}^n + (p - p_\phi k)\,\dot{x} + p_\phi\dot{\phi}\right] + \psi_n(1)\,\psi^n(0)\right\}\Bigg|_{\theta=0}. \tag{24}$$

Then by making the shift

$$p \to P + p_\phi k \tag{25}$$

and by integrating over P, we get

$$G^c = \int \mathrm{d}\phi_a\,\mathrm{d}\phi_b\delta\left(\phi_a - k\cdot x(0)\right)\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{e}^{\mathrm{i}p\cdot(x_b - x_a)} \times \int_0^\infty \mathrm{d}T \mathrm{e}^{\mathrm{i}T\left(p^2 - m^2\right)}\int \mathrm{D}\phi \mathrm{D}p_\phi \times \exp\left\{\mathrm{i}\int_0^T \mathrm{d}\tau\left[p_\phi\left(\dot{\phi} + 2\,(p\cdot k)\right) - h^{\mu\nu}(\phi)p_\mu p_\nu\right]\right\} \times \exp\left(\mathrm{i}\gamma^n \frac{\partial_l}{\partial\theta^n}\right)\mathcal{I}(\phi, T, \theta)\Bigg|_{\theta=0}, \tag{26}$$

where the factor $\mathcal{I}(\phi, T, \theta)$ is given by

$$\mathcal{I}(\phi, T, \theta) = \int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi \exp\left\{\psi_\mu(1)\,\psi^\mu(0) + \int_0^T \mathrm{d}\tau\left[\psi_n\dot{\psi}^n + 2\mathcal{F}^{\mu\nu}(\phi)\,\psi^\mu\psi^\nu\right]\right\}, \tag{27}$$

with

$$\mathcal{F}^{\mu\nu}(\phi) = f'(\phi)\, k^\mu a^{\nu\sigma}p_\sigma = g f'(\phi)\,\mathfrak{f}^{\mu\nu}\,, \tag{28}$$

where g is a numerical factor that we put to 1 in the end of the calculation, and the tensor $\mathfrak{f}^{\mu\nu}$ is given by

$$\mathfrak{f}^{\mu\nu} = k^\mu a^{\nu\sigma}p_\sigma \qquad \mathfrak{f}^2 = 0\,. \tag{29}$$

Let us now do the integration over the Grassmannian variables, to express G^c only through bosonic path integrals. Since the integration variables ψ obey the boundary condition $\psi(0)+\psi(1)=\theta$, it is suitable, in order to calculate $\mathcal{I}(\phi,T,\theta)$, to change ψ by ξ, where

$$\psi=\frac{1}{2}\xi+\frac{\theta}{2}\,. \tag{30}$$

The new variables ξ obey the following boundary condition:

$$\xi(0)+\xi(1)=0\,. \tag{31}$$

Then, in order to obtain a more familiar form with respect to the ξ variables, we change the proper time from τ to σ, where

$$\mathrm{d}\sigma=\frac{1}{T}f'(\phi)\,\mathrm{d}\tau\,. \tag{32}$$

The factor $\mathcal{I}(\phi,T,\theta)$ will then be given through the Grassmann Gaussian integral

$$\begin{aligned}&\mathcal{I}(\phi,T,\theta)\\&=\exp\left(-\frac{g}{2}\tilde{\lambda}\mathfrak{f}_{\mu\nu}\theta^\mu\theta^\nu\right)\int\mathcal{D}\xi\\&\times\exp\left\{\int_0^1\left[\frac{1}{4}\xi_\mu\dot{\xi}^\mu-\frac{1}{2}g\tilde{\lambda}\mathfrak{f}_{\mu\nu}\xi^n\xi^m-g\tilde{\lambda}\mathfrak{f}_{\mu\nu}\theta^\mu\xi^\nu\right]\mathrm{d}\sigma\right\}\Bigg|_{\theta=0},\end{aligned} \tag{33}$$

where

$$\tilde{\lambda}=T\int_0^1 f'(\phi)\,\mathrm{d}\tau\,. \tag{34}$$

Since $\mathfrak{f}_{nm}$ is constant, the problem is brought to the one of a constant electromagnetic field with five-dimensional extension. $\mathcal{I}(\phi,T,\theta)$ can then be evaluated to be

$$\begin{aligned}&\mathcal{I}(\phi,T,\theta)\\&=\det{}^{\frac{1}{2}}\left[\frac{M(g)}{M(g=0)}\right]\exp\left(-\frac{g}{2}\tilde{\lambda}\mathfrak{f}_{\mu\nu}\theta^\mu\theta^\nu\right)\\&\quad\times\exp\left\{\int_0^1\left[\mathcal{J}^\mu(\sigma')\left(M^{-1}\right)_{\mu\nu}\mathcal{J}^\nu(\sigma)\right]\mathrm{d}\sigma'\,\mathrm{d}\sigma\right\},\end{aligned} \tag{35}$$

where the matrix M and the current $\mathcal{J}^\mu$ are given by

$$M_{\mu\nu}(g)=\eta_{mn}\delta'(\sigma-\sigma')-2g\mathfrak{f}_{\mu\nu}\delta(\sigma-\sigma')\,, \tag{36}$$

and

$$\mathcal{J}_\mu=g\tilde{\lambda}\mathfrak{f}_{\mu\nu}\theta^\nu\,. \tag{37}$$

The determinant in (35) can be written as [25, 26]

$$\begin{aligned}&\det\left[\frac{M(g)}{M(0)}\right]\\&=\exp\{\mathrm{Tr}\left[\log M(g)-\log M(0)\right]\}\\&=\exp\left\{-\mathrm{Tr}\int_0^g\mathrm{d}g'\int\mathrm{d}\sigma\int\mathrm{d}\sigma'\mathcal{R}(g';\sigma,\sigma')\,\mathfrak{f}\right\},\end{aligned} \tag{38}$$

where the tensor $\mathcal{R}_{\mu\nu}(g;\sigma,\sigma')$ is given by

$$\mathcal{R}=\left(\frac{1}{2}\eta\varepsilon(\sigma-\sigma')-\frac{1}{2}\tanh\left(g\tilde{\lambda}\mathfrak{f}\right)\right)\exp\left[g\tilde{\lambda}\mathfrak{f}(\sigma-\sigma')\right], \tag{39}$$

and $\varepsilon(\sigma)$ is the sign of σ.

Using now the property

$$\exp\left[\mathrm{Tr}(\ln A)\right]=\det(A)\,, \tag{40}$$

the factor $\mathcal{I}(\phi,T,\theta)$ will be rearranged as follows:

$$\mathcal{I}(\phi,T,\theta)=\det{}^{\frac{1}{2}}\left(\cosh g\tilde{\lambda}\mathfrak{f}\right)\left(1-B_{\mu\nu}\theta^\mu\theta^\nu\right), \tag{41}$$

where the tensor $B_{\mu\nu}$, which has to be understood as a matrix, is given by

$$B=\frac{1}{2}\tanh\left(g\tilde{\lambda}\mathfrak{f}\right). \tag{42}$$

In the case when $k^2=0$, we have $\mathfrak{f}^2=0$. So it is easy to show that

$$B=\frac{1}{2}g\tilde{\lambda}\mathfrak{f} \tag{43}$$

and

$$\mathcal{I}(x,T,\theta)=1+\frac{\tilde{\lambda}}{2}k_\mu a_{\nu\sigma}p^\sigma\theta^\mu\theta^\nu\,. \tag{44}$$

Then the factor containing the Gamma matrices will take the form

$$\begin{aligned}&\exp\left(\mathrm{i}\gamma^n\frac{\partial_l}{\partial\theta^n}\right)\mathcal{I}(x,T,\theta)\Big|_{\theta=0}\\&=\left(1-\frac{1}{2}k_\mu p^\sigma\gamma^\mu\gamma^\nu\int_0^T a_{\nu\sigma}f'(\phi)\,\mathrm{d}\tau\right),\end{aligned} \tag{45}$$

and, consequently, the Green function G^c can be expressed through only bosonic path integrals

$$\begin{aligned}G^c=&\int\mathrm{d}\phi_a\,\mathrm{d}\phi_b\,\delta(\phi_a-k\cdot x(0))\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{e}^{\mathrm{i}p\cdot(x_b-x_a)}\\&\times\int_0^\infty\mathrm{d}T\mathrm{e}^{\mathrm{i}T\left(p^2-m^2\right)}\int\mathrm{D}\phi\mathrm{D}p_\phi\\&\times\left[1-\frac{1}{2}k_\mu p^\sigma\gamma^\mu\gamma^\nu\int_0^T a_{\nu\sigma}f'(\phi)\,\mathrm{d}\tau\right]\\&\times\exp\left\{\mathrm{i}\int_0^1\mathrm{d}\tau\left[p_\phi\left(\dot{\phi}+2(p\cdot k)\right)-h^{\mu\nu}(\phi)p_\mu p_\nu\right]\right\}.\end{aligned} \tag{46}$$

The integration over p_ϕ gives a delta functional $\delta\left(\dot{\phi}+2pk\right)$ that is related directly to the Lagrangian equation of motion projected in the direction of the plane wave. By the vanishing of the argument of $\delta\left(\dot{\phi}+2pk\right)$, we get

$$\mathrm{d}\tau=-\frac{\mathrm{d}\phi}{2pk}\,. \tag{47}$$

Integrating now over the plane wave variable ϕ we obtain the final expression of the Green function G^c:

$$G^c = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{\mathrm{i}p\cdot(x_b - x_a)} \int_0^\infty \mathrm{d}T \mathrm{e}^{\mathrm{i}T\left(p^2 - m^2\right)} \times \left[1 + \frac{k_\mu p^\sigma}{4pk}\gamma^\mu\gamma^\nu \left(h_{\nu\sigma}(k\cdot x_b) - h_{\nu\sigma}(k\cdot x_a)\right)\right] \times \exp\left\{\mathrm{i}\frac{p_\mu p_\nu}{2pk}\int_{k\cdot x_a}^{k\cdot x_b} h^{\mu\nu}(\phi)\,\mathrm{d}\phi\right\}. \tag{48}$$

In order to symmetrize this expression we write

$$\left(1 + \frac{\hat{k}\left(\hat{b} - \hat{a}\right)}{4pk}\right) = \left(1 - \frac{\hat{k}\hat{a}}{4pk}\right)\left(1 + \frac{\hat{k}\hat{b}}{4pk}\right), \tag{49}$$

with $\hat{a} = a\cdot\gamma = a_\mu\gamma^\mu$, and we do the integration over T:

$$G^c = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{\mathrm{i}p\cdot(x_b - x_a)} \left(1 + \frac{k_\mu p^\sigma}{4pk} h_{\nu\sigma}(k\cdot x_b)\gamma^\mu\gamma^\nu\right) \times \frac{1}{p^2 - m^2 + \mathrm{i}\epsilon} \exp\left\{\mathrm{i}\frac{p_\mu p_\nu}{2pk}\int_{k\cdot x_a}^{k\cdot x_b} h^{\mu\nu}(\phi)\,\mathrm{d}\phi\right\} \times \left(1 - \frac{k_\mu p^\sigma}{4pk} h_{\nu\sigma}(k\cdot x_a)\gamma^\mu\gamma^\nu\right). \tag{50}$$

Now, we change p by $-p$ in the last expression of G^c and we incorporate it in (16) to obtain the closed expression of $S^c(x_b, x_a)$

$$S^c(x_b, x_a) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}p\cdot(x_b - x_a)} \left(1 + \frac{k_\mu p^\sigma}{4pk} h_{\nu\sigma}(k\cdot x_b)\gamma^\mu\gamma^\nu\right) \times \frac{\hat{p} + m}{p^2 - m^2 + \mathrm{i}\epsilon} \exp\left\{\mathrm{i}\frac{p_\mu p_\nu}{2pk}\int_{k\cdot x_a}^{k\cdot x_b} h^{\mu\nu}(\phi)\,\mathrm{d}\phi\right\} \times \left(1 - \frac{k_\mu p^\sigma}{4pk} h_{\nu\sigma}(k\cdot x_a)\gamma^\mu\gamma^\nu\right). \tag{51}$$

In order to determine the wave functions, let us integrate over the energy p^0 and employ the projectors of the positive and negative energy states

$$\Lambda_+(p) = \sum_{\pm s} u(p,s)\,\bar{u}(p,s) = \frac{\hat{p} + m}{2m}, \tag{52}$$

$$\Lambda_-(p) = -\sum_{\pm s} v(p,s)\,\bar{v}(p,s) = \frac{-\hat{p} + m}{2m}. \tag{53}$$

We then obtain for $S^c(x_b, x_a)$ the following form:

$$S^c(x_b, x_a) = -\mathrm{i}\theta(t_b - t_a)\int \mathrm{d}^3 p \sum_{\pm s}\psi^{(+)}_{s,\mathbf{p}}(x_b)\,\bar{\psi}^{(+)}_{s,\mathbf{p}}(x_a) + \mathrm{i}\theta(t_a - t_b)\int \mathrm{d}^3 p \sum_{\pm s}\psi^{(-)}_{s,\mathbf{p}}(x_b)\,\bar{\psi}^{(-)}_{s,\mathbf{p}}(x_a), \tag{54}$$

where

$$p^0 = \left(p^2 + m^2\right)^{1/2}, \tag{55}$$

and the wave functions are given by

$$\psi^{(+)}_{s,\mathbf{p}}(x) = \frac{1}{(2\pi)^{3/2}}\left(\frac{m}{p^0}\right)^{1/2}\left(1 + \frac{k_\mu p^\sigma}{2pk} h_{\nu\sigma}(k\cdot x)\gamma^\mu\gamma^\nu\right) \times u(p,s)\exp\left\{-\mathrm{i}p\cdot x - \mathrm{i}\frac{p_\mu p_\nu}{4pk}\int_c^{k\cdot x} h^{\mu\nu}(\phi)\,\mathrm{d}\phi\right\}, \tag{56}$$

and

$$\psi^{(-)}_{s,\mathbf{p}}(x) = \frac{1}{(2\pi)^{3/2}}\left(\frac{m}{p^0}\right)^{1/2}\left(1 - \frac{k_\mu p^\sigma}{4pk} h_{\nu\sigma}(k\cdot x)\gamma^\mu\gamma^\nu\right) \times v(p,s)\exp\left\{\mathrm{i}p\cdot x - \mathrm{i}\frac{p_\mu p_\nu}{2pk}\int_c^{k\cdot x} h^{\mu\nu}(\phi)\,\mathrm{d}\phi\right\}. \tag{57}$$

We remark here that the obtained solutions have a structure similar to the structure of the Volkov solutions corresponding to the electromagnetic plane wave. This is due to the fact that the interaction depends only on the variable $k\cdot x$ and the property $k^2 = 0$.

Also the propagator $S^c(x_b, x_a)$ can be written in the form

$$S^c(x_b, x_a) = \int_0^\infty \mathrm{d}T \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathcal{P}_{\mathrm{SF}}(x_b, x_a) \times \mathrm{e}^{-\mathrm{i}p\cdot(x_b - x_a)}\mathrm{e}^{\mathrm{i}T\left(p^2 - m^2\right)} \times \exp\left\{\mathrm{i}\frac{p_\mu p_\nu}{2pk}\int_{k\cdot x_a}^{k\cdot x_b} h^{\mu\nu}(\phi)\,\mathrm{d}\phi\right\}, \tag{58}$$

where $\mathcal{P}_{\mathrm{SF}}(x_b, x_a)$ is the Polyakov spin factor (SF) [29], which is given, in this case, by

$$\mathcal{P}_{\mathrm{SF}}(x_b, x_a) = \gamma^\mu p_\mu + m + \frac{k_\mu p^\sigma}{4pk} m\left(h_{\nu\sigma}(k\cdot x_b) - h_{\nu\sigma}(k\cdot x_a)\right)\gamma^\mu\gamma^\nu + \frac{p_\alpha k_\mu p^\sigma}{4pk} h_{\nu\sigma}(k\cdot x_b)\left[\gamma^\mu\gamma^\nu\gamma^\alpha - \gamma^\alpha\gamma^\mu\gamma^\nu\right]. \tag{59}$$

The Green function of a scalar particle can then be deduced by separating the spin factor:

$$\Delta^c(x_b, x_a) = \int_0^\infty \mathrm{d}T \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}p\cdot(x_b - x_a)}\mathrm{e}^{\mathrm{i}T\left(p^2 - m^2\right)} \times \exp\left\{\mathrm{i}\frac{p_\mu p_\nu}{2pk}\int_{k\cdot x_a}^{k\cdot x_b} h^{\mu\nu}(\phi)\,\mathrm{d}\phi\right\}. \tag{60}$$

4 Conclusion

In this paper we have given a covariant path integral method to analyze the problem of a Dirac particle subject to a weak gravitational plane wave in $(3+1)$ dimensions. The relative Green function is presented by means of supersymmetric path integrals in the so-called global projection, where the internal motion relative to the spin of the fermion is described by odd Grassmannian variables. The $(3+1)$ dimensional problem is reduced to a $(1+1)$ dimensional one by using the identity (23). Since the pseudoclassical action has a more familiar form with respect to the ψ-variables, we were able to express the Green function only through one-dimensional bosonic path integrals. We have exactly calculated the relative propagator and we have found the wave functions.

Through the formulation given above we conclude that the supersymmetric path integrals are a powerful method to study relativistic one fermion theory.

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